On generalized Hermite polynomials

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1. Introduction

In a previous paper [1], on the solvability, in the sense of Picard-Vessiot theory, of a second order linear differential equation, R. Roberts invented generalized Hermite polynomials. Let m be a positive integer, and let n be a non-negative integer with

$$n \equiv 0 \quad \text{or} \quad n \equiv 1 \mod m+1 . \tag{1}$$

Then the generalized Hermite polynomial $H_{n,m}(x)$, of degree n and of index m, can be defined as the monic polynomial satisfying the differential equation

$$\frac{d^2y}{dx^2} - 2x^m \frac{dy}{dx} + 2nx^{m-1}y = 0 . \tag{2}$$

When $m=1$, (2) reduces to the classical differential equation

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2ny = 0 , \tag{3}$$

and accordingly our $H_{n,1}(x)$ coincides with the Hermite polynomial $H_n(x)$. (Although in some books the highest coefficient of $H_n(x)$ is 2^n, we assume that the Hermite polynomials are monic in this paper.) In this case, (1) does not give any restriction for n.

The main purpose of this paper is to generalize the known completeness of Hermite polynomials in the following form:

Theorem. *Let m be a positive integer. We put*

$$\mathcal{H}_0^{(m)} = \left\{ \exp\left(-\frac{x^{m+1}}{m+1}\right) x^{\frac{m-1}{2}} H_{n,m}(x) ; \quad n=0,\ m+1,\ 2(m+1),\ \cdots\cdots \right\} ,$$

$$\mathcal{H}_1^{(m)} = \left\{ \exp\left(-\frac{x^{m+1}}{m+1}\right) x^{\frac{m-1}{2}} H_{n,m}(x) ; \quad n=1,\ m+2,\ 2m+3,\ \cdots\cdots \right\} .$$

Then each system $\mathcal{H}_i^{(m)}$ $(i=0, 1)$ forms a complete orthogonal system in the open interval $(0, \infty)$.

In order to prove the theorem we shall apply the theory of the eigenvalue problem for ordinary differential equations of the second order, developed mainly by H. Weyl, M.H. Stone, E.C. Titchmarsh,

K. Kodaira and K. Yosida (cf. [2], [3], [4], [5] and [6]), to our differential operator $\mathfrak{L}^{(m)}$ given by

$$2\mathfrak{L}_x^{(m)}y=-\frac{d}{dx}\left(x^{1-m}\frac{dy}{dx}\right)+\left(x^{m+1}-m-\frac{m^2-1}{4x^{m+1}}\right)y\,, \tag{4}$$

which is the generalization of

$$2\mathfrak{L}_x y=-\frac{d^2y}{dx^2}+(x^2-1)y\,, \tag{5}$$

of Hermite polynomials. It is to be noted that because for $m>1$ the coefficients of $\mathfrak{L}^{(m)}$ have poles at 0, we cannot treat the problem directly in the interval $(-\infty, \infty)$.

In the last part of the paper, the generating functions of the generalized Hermite polynomials shall be computed.

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2. Fundamental system of solutions

Let us solve the differential equation

$$\mathfrak{L}_x^{(m)}y=l\,y\,, \tag{6}$$

with the complex parameter l, by means of Kummer's functions, about which the readers may refer e.g. Slater [7].

Let a and b be complex numbers. When b is not a non-positive integer, the power series

$$\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}\frac{z^n}{n!}=1+\frac{a}{b}\frac{z}{1!}+\frac{a(a+1)}{b(b+1)}\frac{z^2}{2!}+\cdots\,. \tag{7}$$

is convergent in the whole plane, and difines the Kummer's function $F(a, b; z)$. When x approaches to ∞ along the real axis, we have

$$F(a, b; x)=\frac{\Gamma(b)}{\Gamma(a)}\exp(x)x^{a-b}\{1+O(x^{-1})\}\,. \tag{8}$$

(Slater [7]), which will be used later.

Let us suppose that b is not an integer. Then the pair of functions

$$\begin{aligned} u_1(z)&=F(a, b; z)\\ u_2(z)&=z^{1-b}F(a-b+1,\quad 2-b; z)\,, \end{aligned} \tag{9}$$

forms a system of solutions of the following equation

$$z\frac{d^2u}{dz^2}+(b-z)\frac{du}{dz}-au=0\,, \tag{10}$$

known as the Kummer's differential equation.

Now, we substitute

$$a=\frac{-l}{m+1}, \qquad b=\frac{m}{m+1}, \qquad z=\frac{2}{m+1}x^{m+1},$$

and

$$u\left(\frac{2}{m+1}x^{m+1}\right)=v(x),$$

in (10), and we have

$$\frac{d^2v}{dx^2}-2x^m\frac{dv}{dx}+2lx^{m-1}v=0. \tag{11}$$

Therefore the pair:

$$\begin{aligned} v_1(x, l)&=F\left(\frac{-l}{m+1}, \frac{m}{m+1}; \frac{2}{m+1}x^{m+1}\right), \\ v_2(x, l)&=-xF\left(\frac{1-l}{m+1}, \frac{m+2}{m+1}; \frac{2}{m+1}x^{m+1}\right), \end{aligned} \tag{12}$$

gives a system of solutions of (11).

Next, we set

$$\rho(x)=\exp\left(\frac{-x^{m+1}}{m+1}\right)x^{\frac{m-1}{2}}, \tag{13}$$

and substitute v in (11) by $\rho^{-1}(x)y(x)$. Then by a direct computation we have

$$-\frac{d}{dx}\left(x^{1-m}\frac{dy}{dx}\right)+\left(x^{m+1}-m-\frac{m^2-1}{4x^{m+1}}-2l\right)y=0,$$

that is

$$\mathfrak{L}_x^{(m)}y=ly. \tag{14}$$

Thus we have found that the pair:

$$\begin{aligned} s_1(x, l)&=\rho(x)v_1(x, l) \\ s_2(x, l)&=\rho(x)v_2(x, l), \end{aligned} \tag{15}$$

is a system of solutions of the equation (14).

For functions $f(x)$ and $g(x)$, we introduce the bracket $[f, g](x)$ by

$$[f, g](x)=x^{1-m}(f(x)g'(x)-f'(x)g(x)). \tag{16}$$

If, in particular, f and g satisfy the same equation $\mathfrak{L}_x^{(m)}y=ly$, then

$[f, g](x)$ does not depend on x. Hence we may write $[f, g]$ for $[f, g](x)$ in this case.

As functions of l, $s_i(x, l)$ and $(d/dx)s_i(x, l)$ $(i=1, 2)$ are holomorphic in the whole l-plane. Moreover, by denoting the conjugate complex number by the bar, we have

$$s_i(x, \bar{l})=\overline{s_i(x, l)} \qquad i=1, 2 . \tag{17}$$

Next,

$$v_1(o, l)=1, \quad v_2(o, l)=0, \quad v_1'(o, l)=0, \quad v_2'(o, l)=-1$$

and

$$\begin{aligned}[s_2, s_1]&=x^{1-m}(\rho v_2(\rho v_1)'-\rho v_1(\rho v_2)')\\ &=x^{1-m}\rho^2(v_2 v_1'-v_2' v_1)\\ &=\exp\left(\frac{-x^{m+1}}{m+1}\right)(v_2 v_1'-v_2' v_1) ,\end{aligned}$$

imply that

$$[s_2, s_1]=1 . \tag{18}$$

Hence $\{s_1(x, l), s_2(x, l)\}$ is a *system of fundamental solutions* of $\mathfrak{L}_x^{(m)}y=ly$ (see Kodaira [5]).

3. Generalized Hermite polynomials and their orthogonality relations

Roberts defined the polynomial $H_{n,m}(x)$ by

$$H_{n,m}(x)=x^n+\sum_{1\le j\le\left[\frac{n}{m+1}\right]}\frac{(-1)^j\prod_{i=0}^{j-1}(n-i(m+1))(n-i(m+1)-1)}{2^j(m+1)^j j!}\times x^{n-j(m+1)} , \tag{19}$$

in [1].

When $n=k(m+1)$, using the formula

$$c(c+1)\cdot\cdots\cdot(c+k-1)=\frac{\Gamma(c+k)}{\Gamma(c)} , \tag{20}$$

we have

$$\prod_{i=0}^{j-1}(n-i(m+1))(n-i(m+1)-1)=\frac{k!}{(k-j)!}\frac{\Gamma\left(k+\frac{m}{m+1}\right)}{\Gamma\left(k-j+\frac{m}{m+1}\right)}(m+1)^{2j}, \tag{21}$$

which follows

$$(22)\quad H_{k(m+1),n}(x)=\sum_{j=0}^{k}\binom{k}{j}\left(\frac{m+1}{-2}\right)^{k-j}\frac{\Gamma\left(k+\frac{m}{m+1}\right)}{\Gamma\left(j+\frac{m}{m+1}\right)}x^{j(m+1)}\ .$$

On the other hand, $v_1(x, k(m+1))=F\left(-k, \frac{m}{m+1}, \frac{2}{m+1}x^{m+1}\right)$ is a polynomial of x^{m+1} of degree k, and a direct computation shows that

$$(23)\qquad v_1(x, k(m+1))=\sum_{j=0}^{k}\binom{k}{j}\left(\frac{-2}{m+1}\right)^{j}\frac{\Gamma\left(\frac{m}{m+1}\right)}{\Gamma\left(j+\frac{m}{m+1}\right)}x^{j(m+1)}\ .$$

Hence we have

$$(24)\qquad v_1(x, k(m+1))=(-1)^k\frac{\Gamma\left(\frac{m}{m+1}\right)}{\Gamma\left(k+\frac{m}{m+1}\right)}\left(\frac{2}{m+1}\right)^k H_{k(m+1),m}(x)\ .$$

In a similar way for $n=k(m+1)+1$, we have

$$(25)\quad \prod_{i=0}^{j-1}(n-i(m+1))(n-i(m+1)-1)=\frac{k!}{(k-j)!}\frac{\Gamma\left(k+\frac{m+2}{m+1}\right)}{\Gamma\left(k-j+\frac{m+2}{m+1}\right)}(m+1)^{2j}\ ,$$

$$(26)\qquad H_{k(m+1)+1,m}(x)=\sum_{j=0}^{k}\binom{k}{j}\left(\frac{m+1}{-2}\right)^{k-j}\frac{\Gamma\left(k+\frac{m+2}{m+1}\right)}{\Gamma\left(j+\frac{m+2}{m+1}\right)}x^{j(m+1)+1}\ ,$$

$$(27)\qquad v_2(x, k(m+1)+1)=(-1)^{k+1}\frac{\Gamma\left(\frac{m+2}{m+1}\right)}{\Gamma\left(k+\frac{m+2}{m+1}\right)}\left(\frac{2}{m+1}\right)^k H_{k(m+1)+1,m}(x)\ .$$

Next, we put $q(x)=\exp\left(\frac{-2}{m+1}x^{m+1}\right)$. Then the equation (11) can be written in the following form.

$$(28)\qquad (qv')'+2l\rho^2 v=0\ .$$

For $v_s=H_{s,m}$ and $v_t=H_{t,m}$, we have

$$(29)\qquad (qv_s')'+2s\rho^2 v_s=0\ ,$$

$$(30)\qquad (qv_t')'+2t\rho^2 v_t=0\ .$$

$(29)\cdot v_t-(30)\cdot v_s$ gives us

$$2(s-t)\rho^2 v_s v_t=(q(v_s v_t'-v_t v_s'))' . \tag{31}$$

Since $v_s v_t'-v_t v_s'$ is a polynomial of x, we have

$$\{q(v_s v_t'-v_t v_s')\}(\infty)=0 .$$

On the other hand, if $s\equiv 0 \mod m+1$ then $v_s'(0)=0$, and if $s\equiv 1 \mod m+1$ then $v_s(0)=0$. Hence integrating the both sides of (31) from 0 to ∞, we have

$$\int_0^\infty \exp\left(\frac{-2}{m+1}x^{m+1}\right)x^{m-1}H_{s,m}(x)H_{t,m}(x)=0 , \tag{32}$$

for $s\neq t$ and $s\equiv t \mod m+1$.

4. Completeness

Let us apply the general theory to our operator $\mathfrak{L}^{(m)}$. We shall retain the notations and the terminologies in Kodaira [5].

Since $s_i(x, l)$ is a continuous function of x in the closed interval $[0, 1]$ we have

$$\int_0^1 (s_i(x, l))^2 dx<\infty \qquad \text{for} \quad i=1, 2 .$$

Hence $\mathfrak{L}^{(m)}$ *is of the limit circle type at* 0.

On the other hand, since

$$F\left(\frac{1}{m+1}, \frac{m+2}{m+1}, z\right)=\sum_{n=0}^{\infty}\frac{1}{n(m+1)+1}\frac{z^n}{n!}>\exp\left(\frac{z}{2}\right),$$

for a sufficiently large positive z, we have

$$|s_2(x, 0)|=\exp\left(-\frac{1}{m+1}x^{m+1}\right)x^{\frac{m+1}{2}}F\left(\frac{1}{m+1}, \frac{m+2}{m+1}; \frac{2}{m+1}x^{m+1}\right)>x^{\frac{m+1}{2}} ,$$

for a sufficiently large positive x. Hence for a positive c

$$\int_c^\infty (s_2(x, 0))^2 dx=\infty .$$

This proves that $\mathfrak{L}^{(m)}$ *is of the limit point type at* ∞. Hence the characteristic function $f_\infty(l)$ is given by

$$f_\infty(l)=-\lim_{x\to\infty}\frac{s_2(x, l)}{s_1(x, l)}=-\lim_{x\to\infty}\frac{v_2(x, l)}{v_1(x, l)}$$

$$=\lim_{x\to\infty}\frac{xF\left(\frac{1-l}{m+1}, \frac{m+2}{m+1}; \frac{2}{m+1}x^{m+1}\right)}{F\left(\frac{-l}{m+1}, \frac{m}{m+1}; \frac{2}{m+1}x^{m+1}\right)} .$$

By (8) we have

$$(33)\qquad f_\infty(l)=\left(\frac{m+1}{2}\right)^{\frac{1}{m+1}}\frac{\Gamma\left(\frac{m+2}{m+1}\right)}{\Gamma\left(\frac{m}{m+1}\right)}\frac{\Gamma\left(-\frac{l}{m+1}\right)}{\Gamma\left(\frac{1-l}{m+1}\right)}.$$

In order to find the characteristic function at 0, we take $w_i(x)=s_i(x, 0)$ $(i=1, 2)$ as the function which determines the boundary condition. Since $[s_2, s_1]=1$ and $[s_i, s_i]=0$, we have

$$(34)\qquad \begin{aligned} f_0^1(l)&=\frac{[w_1, s_2(l)]\,(0)}{[w_1, s_1(l)]\,(0)}=\infty\ ,\\ f_0^2(l)&=\frac{[w_2, s_2(l)]\,(0)}{[w_2, s_1(l)]\,(0)}=0\ . \end{aligned}$$

Hence the characteristic matrices are

$$(35)\qquad \begin{pmatrix} f_\infty(l) & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{f_\infty(l)} \end{pmatrix}$$

corresponding to the boundary conditions. Since $f_\infty(l)$ is meromorphic, we have only discrete point spectra in our eigenvalue problems. Now we shall consider the two cases separately.

(I) Since Γ has no zero, and the only poles of Γ are $0, -1, -2, \cdots$, the only poles of $f_\infty(l)$ are

$$\lambda_k=k(m+1)\qquad k=0, 1, 2, \cdots$$

and since the residue of Γ at k is $(-1)^k/k!$, the residue of $-f_\infty(l)$ at λ_k is given by

$$(36)\qquad R_k^1=\left(\frac{m+1}{2}\right)^{\frac{1}{m+1}}\frac{\Gamma\left(\frac{m+2}{m+1}\right)}{\Gamma\left(\frac{m}{m+1}\right)}\frac{(-1)^k}{k!}\frac{m+1}{\Gamma\left(\frac{1}{m+1}-k\right)}.$$

Hence $\{(R_k^1)^{1/2}s_1(x, k(m+1));\ k=0, 1, 2, \cdots\}$ forms a complete orthonormal system in $(0, \infty)$.

(II) The only poles of $f_\infty(l)^{-1}$ are

$$\mu_k=k(m+1)+1\qquad k=0, 1, 2, \cdots .$$

and the residue of $f_\infty(l)^{-1}$ at μ_k is given by

$$R_k^2=\left(\frac{m+1}{2}\right)^{-\frac{1}{m+1}}\frac{\Gamma\left(\frac{m}{m+1}\right)}{\Gamma\left(\frac{m+2}{m+1}\right)}\frac{(-1)^{k+1}}{k!}\frac{m+1}{\Gamma\left(-\frac{1}{m+1}-k\right)}. \tag{37}$$

Hence $\{(R_k^2)^{1/2}s_2(x, k(m+1)+1);\ k=0, 1, 2, \cdots\}$ sorms a complete orthonormal system in $(0, \infty)$.

5. Generating functions

Let us generalize the known formula

$$\exp\left(xt-\frac{t^2}{4}\right)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n!}t^n, \tag{38}$$

which gives the generating function of Hermite polynomials.

We fix m as before, and introduce the notation

$$\lfloor n=\prod_{0\le j<\left[\frac{n}{m+1}\right]}(n-j(m+1))\,(n-j(m+1)-1), \tag{39}$$

which is a generalization of $n!$. Also we define the function $G_c(x)$ by

$$G_c(x)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1+c)n!}x^n=(-x)^{\frac{-c}{2}}J_c(2\sqrt{-x}), \tag{40}$$

where $J_c(x)$ is the Bessel function of the first kind of order c (see e.g. Yosida [6].)

By (21) we have for $n=k(m+1)$

$$\lfloor n=k!\frac{\Gamma\left(k+\frac{m}{m+1}\right)}{\Gamma\left(\frac{m}{m+1}\right)}(m+1)^{2k}. \tag{41}$$

By a formal computation using (22), we can obtain the following formula.

$$\sum_{k=0}^{\infty}\frac{H_{k(m+1),m}(x)}{\lfloor k(m+1)}t^{k(m+1)}=\Gamma\left(\frac{m}{m+1}\right)\exp\left(\frac{-t^{m+1}}{2(m+1)}\right)G_{\frac{-1}{m+1}}\left(\frac{(xt)^{m+1}}{(m+1)^2}\right). \tag{42}$$

In a similar way we have,

$$\sum_{n=0}^{\infty}\frac{H_{k(m+1)+1,m}(x)}{(k(m+1)+1)}t^{k(m+1)+1} \tag{43}$$

$$=\Gamma\left(\frac{m+2}{m+1}\right)\exp\left(\frac{-t^{m+1}}{2(m+1)}\right)xtG_{\frac{1}{m+1}}\left(\frac{(xt)^{m+1}}{(m+1)^2}\right)$$

by (26) and (27). Thus the following formula gives the generating function of the generalized Hermite polynomials.

$$\sum_n \frac{H_{n,m}(x)}{\lfloor n} t^n$$
$$=\exp\left(\frac{-t^{m+1}}{2(m+1)}\right)\left\{\Gamma\left(\frac{m}{m+1}\right)G_{\frac{-1}{m+1}}\left(\frac{(xt)^{m+1}}{(m+1)^2}\right)\right.$$
$$\left.+\Gamma\left(\frac{m+2}{m+1}\right)xtG_{\frac{1}{m+1}}\left(\frac{(xt)^{m+1}}{(m+1)^2}\right)\right\}. \tag{44}$$

Bibliography

[1] R. M. Roberts, "On the solvability of a second order linear homogeneous differential equation,"

[2] H. Weyl, "Über gewöhnliche lineara Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen," Math. Ann. 68 (1909), s 220-269.

[3] M. H. Stone, Linear transformations in Hilbert space and their applications to analysis, Amer. Math. Soc. (1932).

[4] E. C. Titchmarsh, Eigenfunction expansions associated with second order differential equations, Oxford (1946).

[5] K. Kodaira, "The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices," Amer. J. Math. 71 (1949), p. 921-945.

[6] K. Yosida, Lectures on differential and integral equations, Interscience Publishers (1960).

[7] L. J. Slater, Confluent hypergeometric functions, Cambridge (1960).

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